EXPECTATIONS, BOND PRICES, AND THE TERMIN STRUCTURE OF INTEREST RATES*

Burtorn G. Malkiel


INTRODUCTION

The relation of short-term to long-term yields has often intrigued both economic theorists and investment analysts. This relationship, usually referred to as the *structure* of interest rates, has been characterized by significantly divergent patterns. Despite the considerable attention devoted to the question, rate-structure theory remains in an uncomfortable state of confusion. In addition, an unfortunate lacuna still exists between the writings of economic theorists and the practices of bond traders and institutional investors. This study examines briefly the principle hypotheses which have been offered to explain the relationship of short to long rates. Then, a new theoretical apparatus is offered which, it is hoped, will at once provide a useful framework for amending and reconciling our current body of theory and also bring into closer conformity the beliefs of academic economists and of those close to the debt market.

Traditional doctrine has typically formulated and analyzed the rate-structure problem in terms of a combination of "spot" and "forward" trading.¹ The rate of interest for a two-year loan is conceived as being compounded out of the "spot" rate for loans of one year and the "forward" rate of interest for one-year loans to be executed at the beginning of the second year. Writing the two-year rate (the "long" rate) as \( R_2 \), the current one-year ("short") rate as \( r_1 \), and the forward one-year "short" rate as \( r_2 \), we have \((1 + R_2)^2 = (1 + r_1)(1 + r_2)\). Thus, the system of rates for various maturities can be reduced to the short rate combined with a series of relevant forward short rates.² It is then possible to account for different rate

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* I am greatly indebted to Professors W. J. Baumol and R. E. Quandt, who have offered several useful suggestions and criticisms. The helpful comments of Professors L. V. Chandler and F. Machlup are also gratefully acknowledged.

2. Hicks assumes that all funds are retained in the investment until maturity.
structures. If future short-term rates are expected to fall, then the long-term average of those rates must necessarily be below the current short rate. Similarly, long rates will exceed the current short rate if future short rates are expected to rise. Perhaps the most articulate spokesman for this doctrine, F. A. Lutz, abstracting from costs of investment and institutional rigidities, is able to explain any pattern of rates in this manner.³

The short-rate expectational theory has been beleaguered by critics. Building on the Keynesian theory of "normal backwar- dation" in the futures market, Hicks offered one of the earliest qualifications. Hicks argued that even if short rates are expected to remain unchanged, the forward short rate can normally be expected to exceed the current short rate by a risk premium which must be offered the holder of a bond to compensate him for assuming the risks of price fluctuations.⁴ Thus, the "normal relationship" is for long rates (which are averages of forward short rates) to exceed short rates. Only if the short rate is considered abnormally high could long rates be below short rates.⁵ Other critics have confuted the Lutz theory's basic behavioral postulate, i.e., that investors do in fact decide whether to invest in bonds or bills on the basis of their expectations of future short rates. Investors faced with the choice now of buying either a five- or a ten-year government bond are simply not capable of accurately predicting bill rates from 1967 through 1972. More important, it is doubtful whether any bond investors believe they are capable of doing so. The *reductio ad absurdum* of the argument has been offered in a well-known quip by Mrs. Robinson: If the investor happens to buy consols then he must think he knows exactly what the rates of interest will be every day from today until Kingdom Come.⁶

Recently, the Lutz theory has come under increasingly severe attack. Rather than offering suggestions as modifications, the newer critics have attacked the basic foundations of the theory.⁷ J. M.

⁵ Hicks, *op. cit.*, p. 147.
⁷ A notable exception is J. W. Conard's *Introduction to the Theory of Interest* (Berkeley: University of California Press, 1959), Part Three. Conard makes a valiant attempt to incorporate many of the modifications and criticisms using the Lutz analysis as the cornerstone of an over-all theory of the term structure.
Culbertson lays stress on the institutional impediments to perfect mobility and to changes in the maturity structure of the supply of debt instruments as determinants. But Culbertson's assault is not merely confined to the simplifying assumptions of the Lutz hypothesis; it is directed at the whole structure of the theory. Culbertson finds the expectational explanation theoretically unsatisfactory and doubts that expectations are a major determinant of the term structure of rates. Moreover, Luckett has argued forcefully that once Lutz leaves the world of perfect certainty, awkward logical difficulties arise which are sufficient to throw the entire internal consistency of the theory into question. As a result, a very appealing theory has been thrown under considerable suspicion.

This paper takes the position that the basic expectational approach of the Lutz theory is both correct and of substantial importance in understanding the actual behavior of market interest rates of securities with different terms to maturity. Nevertheless, we assert that the yield-maturity relationship is more clearly perceived when explicit recognition is given to bond prices. We propose to examine rigorously the nexus between market interest rates and bond prices. Then the Lutz theory will be recast in terms consistent with the practices of bond investors and traders. Expectations will be introduced through explicit expected price changes rather than expected future short rates. Finally, we shall examine the implications of relaxing some of the assumptions inherent in the analytical model and of introducing expectations into the supply as well as the demand side of the market.

The Mathematics of Bond Prices

Economists have typically formulated theories of the structure of interest rates in terms of bond yields to the exclusion of bond prices. Keynes probably came closest to an explicit realization of the effect of price risks as a major determinant in the desire for liquidity and, therefore, of interest rates. Even Keynes, however, did not adequately call attention to the precise relationship between changes in bond yields and bond prices. Keynes argued that with a long-term rate of interest of 4 per cent, if it were feared that the rate would

rise faster than 0.16 per cent, per annum, then cash would be preferred to bonds. He reasoned that the fall in bond prices associated with the rise in the interest rate would more than offset the coupon interest received. But if the issue in question were a ten-year (4 per cent coupon) bond, it would take a 0.50 per cent rise in yields to satisfy his argument. For a five-year bond, rates would have to rise by more than 0.91 per cent to wipe out an amount of capital equivalent to the interest received. Keynes was not careful to point out that his analysis is approximately correct only in the case of a perpetual bond. I feel the implication of yield changes on bond prices has usually received inadequate or imprecise attention in the literature. Bonds are traded in terms of price, not yield. They are bought and sold by speculators, long-term investors and financial institutions who are vitally concerned with price movements. We hope to show that a rigorous examination of the nexus between bond yields and market prices can be enormously helpful in understanding the actual fluctuations of yields in the bond markets.

The market value of a bond is determined by four factors: (1) The face value of the bond, i.e., the principal amount to be paid at maturity which we denote by $F$; (2) the coupon or interest paid periodically to the bondholder, denoted by $C$; (3) the effective interest rate per period, $i$, which is referred to as the net return per period or, where we assume annual compounding, the annual yield to maturity; and (4) $N$, the number of years to maturity. The market price, $P$, is simply the sum of the present values of all the coupons to be received as interest and the principal amount to be paid at maturity.

$$P = \frac{C}{(1 + i)} + \frac{C}{(1 + i)^2} + \cdots + \frac{C}{(1 + i)^N} + \frac{F}{(1 + i)^N}.$$  

Summing the geometric progression and simplifying we obtain

$$P = \frac{C}{i} \left[ 1 - \frac{1}{(1 + i)^N} \right] + \frac{F}{(1 + i)^N}.$$  

As $N$ goes to infinity the expression approaches the limit $\frac{C}{i}$. Thus, for a perpetual bond paying $1.00$ per annum, the market value becomes simply the reciprocal of the market rate of interest.

It will be useful to review certain well-known preliminary relationships. Rewriting (2) above as

\[ P = \frac{C}{i} + \frac{(F - C/i)}{(1 + i)^N} \]

and defining the nominal or stated rate of yield \((i_0)\) as

\[ i_0 = \frac{C}{F} \]

we can observe that when the market yield to maturity \((i)\) is equal to the nominal yield then

\[ \frac{C}{i} = F \text{ and } P = F. \] The bond sells at par.

\( \text{(4b)} \) When \( i > i_0 \), then \( \frac{C}{i} < F \) and \( P < F. \) The bond sells at a discount.

\( \text{(4c)} \) When \( i < i_0 \), then \( \frac{C}{i} > F \) and \( P > F. \) The bond sells at a premium.

We may now proceed to examine the relationship between yield changes and bond price movements.

**Theorem 1**: Bond prices move inversely to bond yields.

Proof: Differentiating (1) with respect to \( i \) we obtain

\[ \frac{\partial P}{\partial i} = -\frac{C}{(1 + i)^2} - \frac{2C}{(1 + i)^3} - \cdots \]

\[ -\frac{NC}{(1 + i)^{N+1}} - \frac{NF}{(1 + i)^{N+1}} < 0. \]

**Theorem 2**: For a given change in yield from the nominal yield, changes in bond prices are greater, the longer is the term to maturity.

Proof: We wish to evaluate \( \frac{\partial [P(i) - P(i_0)]}{\partial N} \).

But \( \frac{\partial P(i_0)}{\partial N} = 0 \) since for \( i = i_0, \ P = F, \) a constant, by (4a).

Therefore, it is sufficient to evaluate the sign of \( \frac{\partial P(i)}{\partial N} \). Rewriting (2) we have

\[ P = \frac{C}{i} - (1 + i)^{-N}\left[\frac{C}{i} - F\right]. \]

Differentiating with respect to \( N \) we obtain

\[ \frac{\partial P}{\partial N} = \left[\frac{C}{i} - F\right][1 + i]^{-N}[ln(1 + i)]. \]
When \( i < i_0 \), the bond sells at a premium, \( \left[ \frac{C}{i} - F \right] > 0 \) from (4c) and therefore \( \frac{\partial P}{\partial N} > 0 \). Therefore, if the market yield is below the nominal yield, \( i_0 \), the price of the bond \( P \) will be higher the longer the time to maturity.

When \( i > i_0 \), the bond sells at a discount, \( \left[ \frac{C}{i} - F \right] < 0 \) from (4b) and therefore \( \frac{\partial P}{\partial N} < 0 \). Thus if the market yield is above the nominal yield, the price of the bond \( P \) will be lower as the length of time to maturity increases. Since the absolute and percentage change in bond prices is measured by the difference between the derived \( P \) and \( F \) (\( F = 100 \)) we find that bond-price movements are amplified as time to maturity is increased.

**Theorem 3:** The percentage price changes described in Theorem 2 increase at a diminishing rate as \( N \) increases.

Proof: Differentiating (6) with respect to \( N \) we obtain

\[
\frac{\partial^2 P}{\partial N^2} = - \left[ \frac{C}{i} - F \right] \ln(1 + i)^2 [(1 + i)^{-N}].
\]

Repeating the same argument as above we have for \( i \) below the nominal rate, \( i_0 \), \( \frac{\partial^2 P}{\partial N^2} < 0 \). The percentage price rise from par increases with \( N \) but at a diminishing rate. Similarly, for \( i \) above \( i_0 \), \( \frac{\partial^2 P}{\partial N^2} > 0 \). The percentage price decline from par increases with \( N \) at a diminishing rate.

**Theorem 4:** Price movements resulting from equal absolute (or, what is the same, from equal proportionate) increases and decreases in yield are asymmetric; i.e., a decrease in yields raises bond prices more than the same increase in yields lowers prices.

Proof: We will show that in order to prove asymmetry it is sufficient to show that \( \frac{\partial^2 P}{\partial i^2} > 0 \). Differentiating (5) with respect to \( i \) we have

\[
\frac{\partial^2 P}{\partial i^2} = \frac{2C}{(1 + i)^3} + \frac{2.3C}{(1 + i)^4} + \cdots + \frac{N(N + 1)C}{(1 + i)^{N+2}} + \frac{N(N + 1)F}{(1 + i)^{N+2}} > 0.
\]
Thus the slope of the function $P = f(i)$ becomes less negative as $i$ increases. Consequently, for any $i'$, the average slope of the function over the range between $i'$ and $(i' - \Delta i)$ is steeper than over the range between $i'$ and $(i' + \Delta i)$. Therefore, a (say 10 per cent) increase in yield will result in a smaller absolute and percentage price decline than a 10 per cent decrease in yield will raise bond prices.

**Theorem 5:** The higher is the coupon carried by the bond, the smaller will be the percentage price fluctuation for a given change in yield except for one-year securities and consols.

Proof: We wish to prove \( \frac{\partial \left[ \frac{\partial P}{\partial i} \cdot \frac{i}{P} \right]}{\partial C} > 0 \), for all finite $N \geq 2$.

Differentiating (2) with respect to $i$ we obtain:

\[
\frac{\partial P}{\partial i} = \frac{-C(1 + i)^{N+1} + C(1 + i + Ni) - FNi^2}{i^2(1 + i)^{N+1}}.
\]

Multiplying through by $i/P$ (where we use expression (2) for $P$) and simplifying we obtain

\[
\frac{\partial P}{\partial i} \cdot \frac{i}{P} = \frac{-C(1 + i)^{N+1} + C(1 + i + Ni) - FNi^2}{C(1 + i)^{N+1} - C(1 + i) + Fi(1 + i)}.
\]

Let us write $\Delta(N) = (1 + i)[C(1 + i)^N - C + Fi]^2$ so that

\[
\frac{Fi}{(1 + i)[C(1 + i)^N - C + Fi]^2} = \frac{Fi}{\Delta(N)} > 0 \quad \text{for all finite } N.
\]

Differentiating (9) with respect to $C$ we obtain:

\[
\frac{\partial \left[ \frac{\partial P}{\partial i} \cdot \frac{i}{P} \right]}{\partial C} = \frac{Fi}{\Delta(N)} \left[ 1 + i + (1 + i)^N(Ni - 1 - i) \right].
\]

When $N = 1$, (11) = 0. (11) is positive for all finite $N \geq 2$ as will now be shown by induction. Write

\[
\Phi(N) = 1 + i + (1 + i)^N(Ni - 1 - i).
\]

First note that for $N = 2$, $\Phi(N) = (i^2 + i^3) > 0$ for all $i > 0$. We now prove that $\Phi(N)$ is an increasing function of $N$ so that since $\Phi(N) > 0$ for $N = 2$, $\Phi(N) > 0$ for $N > 2$.

\[
\Phi(N + 1) = 1 + i + (1 + i)^N(1 + i)(Ni - 1), \text{ i.e.,}
\]

\[
\Phi(N + 1) = \Phi(N) + Ni^2(1 + i)^N
\]

\[\therefore \Phi(N + 1) > \Phi(N) \text{ for all } N > 0, i > 0.\]
Thus we conclude

(12) $\phi(N) > 0$ for all finite $N \geq 2$. Hence, since (11) can be
written $\frac{F_i}{\Delta(N)} \cdot \phi(N)$, we conclude by (10) and (12) that

$$\frac{\partial}{\partial C} \left[ \frac{\partial P}{\partial i} \cdot \frac{i}{P} \right] > 0. \quad \text{Q.E.D.}$$

But note that when $N \to \infty$, $\frac{F_i}{\Delta(N)} \to 0$ and (11) $\to 0$.

Table I summarizes several of the relationships described by the
theorems. The greater absolute and diminishing marginal volatility
of long-term bond prices is patently revealed.

A two-year bond is shown to rise or fall about twice as far from
par as a one-year bond. Similarly, a four-year bond fluctuates
almost twice as much as a two-year security. A sixty-four year bond
will not, however, fluctuate in price significantly more than a thirty-
two-year bond, particularly for an increase in yield. Thus, the im-
lications for an investor of extending the maturity of his bond
portfolio can be quite different depending on the maturity sector of
the curve to which it is applicable.\footnote{Product.} Furthermore, an asymmetry of
price movements on either side of the axis from par is revealed.
There is a natural cushion which exists simply in the mathematics of
bond prices which limits a long-term bond's price decline as yields
rise.\footnote{Product.}

4. Frederick R. Macauley has argued that years to maturity is a very
inadequate measure of the true length of a loan. It tells only the date of
the final payment and nothing about the size and frequency of all intermediate
payments. In the case of a very long-term bond, the importance of the ma-
turity date may be so small as to be negligible. Macauley uses the term "dura-
tion" to describe the true length of the bond. The duration of any loan is simply
the weighted average of the maturities of the individual loans that correspond
to each future payment. The present values of the individual payments are used
as weights. As bonds lengthen in time to maturity, true length or duration in-
creases at a decreasing rate. A twenty-five-year bond is surprisingly little dif-
f erent from a fifty-year bond, but a six-year bond is approximately twice as
long as a three-year issue. This is consistent with and helps explain the actual
price relationships described by our theorems. Cf. Frederick R. Macauley, Some
Theoretical Problems Suggested By The Movements of Interest Rates, Bond Yields
and Stock Prices in the United States Since 1856. (New York: National Bureau
of Economic Research, 1938), pp. 44–53.

5. Another important factor is the tax implication of bond discounts. The
effective after-tax yield to maturity is much larger for a discount bond selling
at a given yield to maturity than for a bond selling at par with the same pre-
tax yield. In the case of the discount bond, part of the yield to maturity is taxed
at preferential capital-gains rates. In addition, the remoteness of possible call
features lends added attraction to deep-discount bonds.
### TABLE I

**Selected Price Data for a Bond with Nominal Yield**

(i₀) 3 Per Cent

<table>
<thead>
<tr>
<th>Years to Maturity (N)</th>
<th>Price (P) to yield 4% (i) to maturity</th>
<th>Loss incurred if market yields rise from i₀ to 4% (%)</th>
<th>Marginal loss incurred by extending maturity one additional year (%)</th>
<th>Price (P) to yield 2% (i) to maturity</th>
<th>Gain realized if market yields fall from i₀ to 2% (%)</th>
<th>Marginal gain realized by extending maturity one additional year (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>99.029</td>
<td>.971</td>
<td>.971</td>
<td>100.985</td>
<td>.985</td>
<td>.985</td>
</tr>
<tr>
<td>2</td>
<td>98.096</td>
<td>1.904</td>
<td>.933</td>
<td>101.951</td>
<td>1.951</td>
<td>.966</td>
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<tr>
<td>4</td>
<td>96.337</td>
<td>3.663</td>
<td>.862</td>
<td>103.826</td>
<td>3.826</td>
<td>.928</td>
</tr>
<tr>
<td>8</td>
<td>93.211</td>
<td>6.789</td>
<td>.736</td>
<td>107.359</td>
<td>7.359</td>
<td>.857</td>
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<tr>
<td>16</td>
<td>88.266</td>
<td>11.734</td>
<td>.536</td>
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<td>13.635</td>
<td>.731</td>
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<tr>
<td>32</td>
<td>82.039</td>
<td>17.961</td>
<td>.285</td>
<td>123.551</td>
<td>23.551</td>
<td>.531</td>
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<tr>
<td>64</td>
<td>76.982</td>
<td>23.018</td>
<td>.080</td>
<td>136.009</td>
<td>36.009</td>
<td>.281</td>
</tr>
<tr>
<td>consol</td>
<td>75.000</td>
<td>25.000</td>
<td>.000</td>
<td>150.000</td>
<td>50.000</td>
<td>.000</td>
</tr>
</tbody>
</table>

1. All examples assume semiannual compounding.
An examination of these relationships alone brings us immediately to an important conclusion. Empirical investigations of the shape of the yield curve\(^6\) show that the curve invariably flattens out as term to maturity is extended. We have shown that the price risks inherent in holding long-term bonds over a wide range of maturities are roughly similar. Therefore, it is not surprising that these bonds, which are mathematically almost equivalent securities in terms of potential price fluctuations, will sell in the market at roughly similar yields. Conversely, in the short and intermediate areas of the yield curve, the extension of maturity even for a few years implies considerably different price risks and opportunities for capital gains. Thus it is reasonable to expect that these areas of the yield curve are the ones which will exhibit the more dramatic responses to changes in expectations.\(^7\) We shall show this result specifically in our analytical models which follow.

**Expectations and the Term Structure of Rates**

We now approach the very difficult area of expectations. Fortunately, our *modus operandi* enables us to move gingerly at first. We shall begin by introducing the theoretical construct “the expected normal range of interest rates.” This range will be defined in terms of the level rather than the structure of rates. Our problem is one of finding some correspondence between our *ex ante* theoretical construct and the observable *ex post* empirical data at our disposal, i.e., the historical level of rates.\(^8\) As a first approximation, the aggregate of individuals comprising the “market” will be assumed to believe that the historical range of interest rates will prevail in the future. We shall assume, in the examples which follow, that the expected normal range for government bonds is roughly between 2 per cent and 5 per cent.

First, let us suppose that interest rates for securities of all maturities are fixed at 4\(\frac{1}{2}\) per cent by an arbitrary decree of a *deus ex machina*. For simplicity, we shall consider that all bonds carry coupons of 4\(\frac{1}{2}\) per cent and, therefore, the prices of all securities are fixed at par. Now, in stages, we shall examine the effects of removing completely all controls from the market. In the first stage, we shall

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6. A graphic device used for examining the relationship between the yield and term to maturity of comparable debt securities.

7. It is interesting to compare the above explanation with that offered by Lutz. To form a “shoulder,” short rates would have to be expected to change in the near future and then reach a certain level where they would stay constant.

assume the "market" has formed no expectation as to whether bond
prices will rise or fall. However, in consonance with our construct of
the normal range, investors are well aware of the fact that this level
of rates leaves less to be feared than hoped. The worst situation that
could confront an investor is for him to suffer a rise in interest rates
to 5 per cent. On the other hand, rates could fall to 2 per cent and
still lie within the normal range. Historical interest rate movements
would suggest, however, that such a drastic drop in the level of rates
has been associated with secular rather than cyclical movements.
Thus, investors could not reasonably anticipate that the lower end
of the normal range would be realized within, say, one year. Under
these circumstances we might believe that a plausible one-year normal
range might be between 5 per cent and 3½ per cent. Any precise
figures are really irrelevant. We simply wish to explore the effects of
a subjective expectation of a normal range on the structure of rates.
The critical assumption is that the investor has more to gain than to
fear. Therefore, whatever the expected normal fluctuations for the
coming year may be, our assumption demands that the "market"
believe that these will be contained in a range where the lower limit
is further from 4½ per cent than the upper limit. Again we must
remind our reader, however, that investors have collectively made
no judgment as to the likelihood of interest rates in general moving
up or down.

Columns 2 through 5 of Table II summarize the resulting price
action of bonds of varying maturities if yields should move to the
extremities of the one-year normal range in one year.9 Thus, in-
vestors collectively are faced with the matrix consisting of the array
of percentage gains and losses (negative gains) contained in columns 3
and 5, which we can call states of nature A and B. We next assume,
as a first approximation, that utility is linear in gains. Furthermore,
we postulate that investors' preferences can be represented by a
von Neumann-Morgenstern utility index. We may now choose that
particular relationship where the utility numbers are made equal to
the gains.1 There are at least two grounds upon which one could

9. Note that we have substituted a one-year outlook for the long-run hori-
zon implied by the Lutz theory. Expectations, to the extent that they are made
concerning the likelihood of interest rate changes, are often very vague beyond
the near future. Moreover, the one-year horizon is consistent with the practices
of institutional bond investors.

1. Cf. John von Neumann and Oskar Morgenstern, Theory of Games and
index chosen has minus 100 as its zero point. Minus 100 is the worst possible
outcome. The investor cannot lose more than his investment. No such limitation
exists for his gains.
quarrel with this assumption. In the first place, it is doubtful that utility is linear in money. Second, and in addition to the diminishing utility of money payoffs, it is well known that losses are especially unpleasant. Thus on both counts losses will be more negatively preferred than equivalent gains are desirable, particularly for institutional investing officers. We shall examine later the implications of relaxing this assumption of linearity but, as an initial approximation, we feel justified in considering that the unadjusted gains and losses make up the game matrix facing our collective investor.

We may now proceed to consider this situation as a classical example of a game against nature. The investor must choose among several acts (i.e., the array of maturities available to him) where the desirability of each act depends on the state of nature which will prevail. We have postulated that the decision-maker believes that one state of nature (i.e., either $A$ or $B$) will exist, but he does not know the relative probabilities of each. The standard literature on decision theory suggests a group of criteria to deal with these types of problems.\(^2\) The theoretical justification for using any one of these is shaky. We shall select the Laplace principle of insufficient reason to solve the game matrix.\(^3\) This criterion asserts that since the decision-maker is ignorant of the probabilities of the different possible states of nature, he should treat them all as equally likely. He will then assign to each act $A_i$ its expected utility index

$$U_{i1} + \cdots + U_{in},$$

and choose that act with the largest index. In our case, which has only two states of nature, this is equivalent to attaching a probability of .50 to each state.

Unfortunately, from an empirical point of view, this principle is confronted with serious difficulties. There is really an infinite number of states of nature which could be regarded as "equally likely." For example, yields might remain unchanged, rise 0.5 per cent, fall 0.5


3. The Hurwicz solution which utilizes a subjective "optimism-pessimism index" is inconsistent with our assumption that our representative investor has made no judgment as to the likelihood of interest rates moving up or down. Applying the Wald "maximin" criterion is tantamount to saying that normal backwardation will always force the investors to buy the shortest security available if there exists any possibility of loss. Similarly, the Savage "minimax regret" criterion would lead investors to confine their purchases toward the long end of the yield curve. These latter two criteria may be rejected on empirical grounds as not being in accord with the normal behavior of bond investors.
per cent, etc. Why should the extremes of our one-year normal range be singled out as the natural parametrization of the states for which this criterion is appropriate? Our answer is simply that for our purposes it makes no difference which states of nature we select so long as they conform to our critical assumption which demands that for states to be equally likely they must leave the investor more to hope than fear. By selecting the extremes of the possible range of price movements we achieve the same result as would be obtained by using the uniform distribution over the entire range. The narrower is the expected range of fluctuations, the smaller will be the derived yield differentials. But the differences in results will be only differences in degree. If, however, investors believe that there will be no range of fluctuation during the coming year then the derived yield curve would be horizontal which is precisely what we would expect.

We may now proceed quickly to some results. Column 6 of Table II presents the mathematical expectation of gain for each maturity. Longer-term securities are clearly more desirable to our collective investor than short-term issues. Hence the bond market will not remain at the equilibrium which was formerly imposed upon it by decree. Assuming that the prices of long-term securities are bid up to remove the differences in opportunities for gain among maturities, the prices listed in Column 7 will be realized in the market. At these prices, the derived yields to maturity listed in Column 8 will represent the new equilibrium term structure of rates. There will be no net tendency to shift from one maturity to another from a partial equilibrium-comparative statics point of view.

4. One additional element serving to dampen potential price fluctuations of short-term securities should be noted. The passage of time changes the maturity of the security and thus diminishes the issue's characteristic price fluctuations. The passage of one year cuts the potential volatility of a two-year bond approximately in half.

5. There are, of course, several ways in which the market could equalize the mathematical expectation of gain among maturities. We shall return to this point.

6. We should actually equalize the (discounted) value of the expected price appreciation (depreciation) and the coupons to be received during the holding period, all as a percentage of the purchase price paid for the bond. In our examples we have eliminated the entire mathematical expectation of gain (loss) but the value of the two coupons as a percentage of the market price of the securities will not be equalized. Our approximation was used to permit simplicity of computation. The differences in the derived yield are generally in the order of magnitude of .01 per cent and therefore have no effect on our results.

7. We must assume, however, that investors' expectations as to what constitutes the normal one-year range are tied to the level of the one-year short rate and are, therefore, unchanged. Otherwise, the structure of rates derived after the process of adjustment is likely to set up changed expectations concerning the one-year normal range and further adjustments would be required. This restriction is not, in general, necessary, as we shall later show.
### TABLE II
**Assumed 4\(\frac{1}{4}\) Per Cent Bonds**

<table>
<thead>
<tr>
<th></th>
<th>2</th>
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<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Market price of bond if interest rates rise to 5% in one year(^1)</td>
<td>Resulting price appreciation (%)</td>
<td>Market price of bond if interest rates fall to 3(\frac{1}{2})% in one year</td>
<td>Resulting price appreciation (%)</td>
<td>Mathematical expectation of gain (p: A, B) (p = .50)</td>
<td>Market prices equalizing mathematical expectation of gain (Col. 6) among maturities</td>
<td>Derived structure of rates from Col. 7.</td>
<td>Derived structure of rates (p: A, B) (p = .25)</td>
</tr>
<tr>
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<td>100.13</td>
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<td>+2.04</td>
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<td>109.33</td>
<td>4.12</td>
<td>3.78</td>
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</table>

1. I.e., Price in one year to yield 5 per cent from "present plus one year" to maturity.
2. In one year a one-year security matures at par.
therefore, explained a descending term structure of interest rates when the level of rates is near the upper limit of its normal range. Expectations have been introduced only to the extent of assuming that the normal range of the past will also continue into the future. Our major tool has been an analysis of the mathematics of bond prices. We find that an analysis of potential market-price changes can provide important insights into the behavior of yields of securities with different terms to maturity.

Having started with almost complete uncertainty as the foundation of our analytical model, it is now incumbent on us to introduce expectations in a less passive manner. We suggest that if interest rates appear to be high relative to the normal range, investors may also attach a higher subjective probability to that state of nature in which interest rates fall. This transforms our game matrix from a game against nature into the domain of decision-making under risk. We can now associate with state of nature $A$, the probability $p$, and with state $B$, the probability $(1 - p)$. The new utility index becomes $U_{iA}(p) + U_{iB}(1 - p)$ and again the decision-maker will choose that act having the maximum utility index. If we assume that the “market” attaches a probability of .75 to the state of nature in which rates fall, Column 9 presents the new derived yield-maturity curve. We notice that the expectations introduced have made the curve more sharply descending than before. Nevertheless, the new curve shows the same tendency to level out as term to maturity is extended.

Table III describes a similar case starting from an initial fixed interest rate level of $2\frac{1}{2}$ per cent for all maturities. All bonds are assumed to bear a $2\frac{1}{2}$ per cent coupon. Analogously to our first example, the one-year normal range is postulated to be between 2 per cent and $3\frac{1}{2}$ per cent. In this case, investors have more to fear than hope. The result (Column 8) is an ascending structure of rates which becomes flat for long-term maturities. One curious result should be noted. We find that our theoretical apparatus indicates that a consol should yield less than a fifty-year dated issue despite the fact that the general shape of the yield curve is ascending. This is not a peculiarity of the particular numerical example chosen. The same result will be obtained for any $p$, $0.01 \leq p \leq 1$, i.e., except where there is essentially no prospect for gain. This “anomaly” results solely from the mathematics of bond prices. Furthermore, our finding is consistent

8. This relationship can be explained with the aid of Theorem 4: Note that the asymmetry of bond price movements provides a natural cushion limiting a consol’s potential price decline whereas no such restraint exists to dampen the
### TABLE III
**Assumed 2\(\frac{1}{2}\) Per Cent Coupon Bonds**

<table>
<thead>
<tr>
<th></th>
<th>2</th>
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<th>7</th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Market price of bond if interest rates fall to 2% in one year</td>
<td>Resulting price appreciation (%)</td>
<td>Market price of bond if interest rates rise to 3(\frac{1}{2})% in one year</td>
<td>Resulting price appreciation (%)</td>
<td>Mathemathical expectation of gain ((p: A, B))</td>
<td>Derived structure of rates (from Col. 7)</td>
<td>Derived structure with expectation of greater fluctuations in short rates</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>100.00</td>
<td>A</td>
<td>100.00</td>
<td>B</td>
<td>—</td>
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<td>1(\frac{1}{2})</td>
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<td>99.51</td>
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<td>-3.70</td>
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<td>76.65</td>
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<td>86.38</td>
<td>3.03</td>
<td>3.12</td>
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<td>-15.20</td>
<td>84.80</td>
<td>2.95</td>
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</table>
with the empirical evidence. British consols have typically sold to yield less than long-term dated issues irrespective of the general shape of the yield curve.\footnote{We do not suggest that this can be entirely explained by the mathematics of differential potential price actions. Undoubtedly another powerful determinant is the greater protection from call offered by the deeper discount.} It is one of the strengths of our theory that it is consistent with some of the more bizarre relationships of the yield curve.

Let us now push our expectational analysis one step further. Empirical evidence discloses that short-term interest rates have exhibited a greater volatility than long-term rates. Indeed, our theoretical apparatus indicates that this must occur because investor expectations will cause an ascending curve to be formed when interest rates are low relative to the historical range and a descending curve when rates are relatively high. Therefore, we are neither begging the question nor guilty of circular reasoning if we include in our expectations the belief that long-term interest rates will fluctuate less than do short rates. Thus, instead of assuming that 2 per cent will be the lower limit to which all interest rates can fall, we could hypothesize that short rates would fall .75 per cent to 1.75 per cent but long rates would decline only .25 per cent to 2.25 per cent. Similarly, for the upper limit we shall allow short rates to rise to 4 per cent while long rates rise to 3.50 per cent. Intermediate rates would be scaled accordingly between the two limits. All other assumptions remain identical with those in our previous illustrations. We find in Column 9, Table III, that an ascending yield curve is again formed, but in this case it traces a more sharply upward course than in our previous example. This is an important qualification to the expectational theory. To the extent that institutional rigidities lead to relatively greater volatility in short rates, it is not true that the activities of arbitrageurs will necessarily work in the direction of smoothing these additional yield fluctuations. It is far more likely that the market will recognize these tendencies and that these expectations will tend to be self-fulfilling.\footnote{Of course, we have implicitly assumed in our analytical model that sufficient flexibility exists in the maturity needs of different investors to make arbitrage possible. We do not disagree that institutional impediments to perfect mobility do exist to a sufficient degree that any expectational theory must be modified. Nevertheless, the considerable amount of interchangeability between the maturity sectors which exists on both the demand and supply sides of the consol’s possible price appreciation. (Cf. Columns 3 and 5 for a fifty-year and perpetual bond.) Thus, even though the mathematical expectation of loss (Column 6) is greatest for a perpetual bond, it is not so great as Theorem 2 would require to equalize the derived yields of Column 8 (cf. Columns 4 and 7 for a fifty-year and perpetual bond.)}

\footnote{We do not suggest that this can be entirely explained by the mathematics of differential potential price actions. Undoubtedly another powerful determinant is the greater protection from call offered by the deeper discount.}
ALTERING SOME OF THE ASSUMPTIONS

Let us now examine the effects of altering certain of the explicit and implicit assumptions contained in our analysis. We have alluded to some of these in passing. First, let us assume that the investors’ utility functions weigh losses more heavily than equivalent gains. This will have the effect of making the slope of ascending yield curves steeper and the slope of the descending curves flatter. The basic shapes of the curves in our examples will, in general, not be changed even by making fairly substantial adjustments in the game matrix.² This does suggest, however, that it is somewhat more difficult for descending curves to be formed and may be offered as a partial explanation for the somewhat greater number of ascending curves which are found in the historical data. Thus we find that the Keynes-Hicks analysis easily can be incorporated into our theoretical apparatus. We suggest, however, that the hypothesis of a “normal relationship” can be supported neither by our analytical model nor by the empirical evidence.

Next, we might relax the assumption of the invariance of the normal range. Is it not plausible that investors’ expectations will be influenced to a greater degree by the more immediate past? Thus, when interest rates have been low for a considerable period of time, investors will come to reduce their idea of a normal range. The effect of this suggestion is to make it possible for expectations to form a descending curve at a lower level of interest rates (when rates in the immediate past have been particularly low) than would be possible had proximate rates been at a higher level. Thus, for example, the yield curve on governments in late 1957 descended gently at a level of interest rates of about 3.90 per cent. Yet, during the early summers of 1960 and 1961 when the level of rates was again at 3.90 per cent, the curve was either relatively flat or rising. One difference was that investors in 1960 and 1961 had become accustomed to a significantly higher level of interest rates and, therefore, their expectations as to what would constitute a normal range of rates were considerably changed from those extant in 1957.³
debt market can be easily underestimated. Even if the maturity needs of many classes of financial institutions are rigid, maturity indifference on the part of a small number of investors and issuers may be sufficient to prevent serious market segmentation, and independence in the behavior of different segments of the yield curve.

2. For example, if in the game matrix presented in Table II we double the negative utility values of all losses, we still obtain a descending yield curve which falls from 4.50 per cent to 4.31 per cent (for \( p = .50 \)).

3. Expectational theories have often been criticized on the basis of an empirical analysis of holding-period yields. We have derived hypothetical rate
In our examples we have employed only two illustrative values of \( p, .50 \) and \( .75 \). If investors attach a probability greater than \( .75 \) to the possibility that bond prices will rise (fall), a more sharply descending (ascending) curve will result. And if coupons on existing maturities are not all identical, we can expect differing price reactions, all of which will obey Theorem 5. These price reactions will then alter the yield-maturity pattern, for bonds will now be more or less attractive to the investor depending upon their coupons. This is one reason for the many kinks which exist in the yield curve for government securities.\(^4\) Only a model cast in terms of price changes can deal effectively with this type of adjustment. The introduction of considerations such as call features and the special tax advantages of discount bonds strengthens our analytical findings. When interest rates fall to low levels, many long-term bonds rise in price to where they sell at premiums over their call prices. This diminishes their attractiveness and tends to raise the yields of these long-term securities. Alternatively, high interest rates and accompanying low bond prices free the bondholder from call risks and deep bond discounts lend added attraction because of their higher effective after-tax yields.

Finally, let us consider alternative methods of equalizing the mathematical expectation of gain among securities of various maturities. In our examples, we equalized opportunities for gain by eliminating them, i.e., we allowed all bond prices to rise or fall so that the entire expectation of differential gain was removed. It is interesting to examine what would happen if investors sold shorts and bought structures assuming that investors will act in such a way that the mathematical expectations of gain or loss are equalized among maturities.

But even if an expectational analysis constructed in this manner could provide a complete explanation of the actual rate structure in the market, empirical investigations would not, in general, show that holding-period yields are identical for securities of different terms to maturity. Only in the unusual event that yields move exactly to the limits set by the mathematical expectation of our collective investor will holding-period yields be equalized. Since the mathematical expectation is compounded out of several variables, including individual attitudes toward risk aversion, past interest rates, expected future rates, etc., a resulting equality of holding-period yields should hardly be expected.

\(^4\) The reader is cautioned to consider that even for governments, the most similar group of securities, there is a myriad of special features inherent in individual issues which may also alter their yield relationship, e.g., optional maturities on many issues differentiate them in the minds of the buyers. Other securities may be "put" to the Treasury at the option of the holder. Some issues are partially tax exempt. Most, when owned by a decedent and part of his estate, are redeemable at par for payments of federal estate taxes irrespective of the current market price. Finally prices of maturing issues are often influenced by possible exchange privileges at call or maturity.
longs when prices were expected to rise. In this case short-term bond prices would fall and long-term prices would rise less than in our examples. The resulting level of rates would be higher than in our example solution. Furthermore, it can be shown that the yield curve would become steeper because even small price declines in the short-term area would have a greater effect in raising short yields than the smaller increases in price in the long-term area would tend to raise long yields.  

We have no quarrel with the position taken by Culbertson and others that changes in the maturity structure of the supply of debt instruments are important determinants of the rate structure. One should not expect that arbitrage would neutralize their effect. It is useful, however, to look behind these changes, to examine the possible role which expectations may play in causing those disturbances which are claimed to impede expectations on the demand side from determining the shape of the yield curve. An analysis of the introduction of expectations to the supply side of the market is completely analogous to our previous argument. If issuers of securities believe that interest rates are relatively high compared with their expectations of what constitutes a normal range, they will tend, to whatever extent possible, to issue short-term securities rather than longer bonds. Conversely, if rates appear attractive, issuers will take advantage of the opportunity and issue long-term securities. The motivation of issuers cannot be cast in terms of price risks but must rather be explained by considering the desire to minimize long-run financing costs. Most Secretaries of the Treasury have placed the objective of minimizing the cost of debt service among the primary goals of prudent debt management. Similarly, corporate treasurers are keenly aware of the cost of debt capital and attempt to time their trips to the long-term

5. When we allow alternative methods of equalizing mathematical expectations of gains and losses we need no longer assume that the one-year normal range is tied to the level of the one-year short rate (cf. footnote 7, p. 209). If investors are allowed to arbitrage by selling shorts and buying longs when yields were expected to fall, then the level of rates (assuming it is a function of both long and short rates) may not be changed after the adjustment. Short rates will be higher but long rates will be lower so that the one-year normal range may remain unchanged and therefore no further adjustment is required. If we assume that the level of rates (and, therefore, the one-year range) does change, we cannot a priori tell whether it will be higher or lower and, consequently, it is impossible to make any definite statement about the secondary adjustments in the term structure which will occur.

6. As we have mentioned, impediments to perfect arbitrage do exist. Moreover, the actual or potential increase in supply itself changes the expected normal range for any maturity area.
capital market to avoid, insofar as possible, periods of high interest rates.\footnote{Curiously, the desire to minimize interest costs may not be their most important motivation. There is also a definite stigma attached to the financial reputation of corporations which have included in their capitalizations, securities which bear such high interest coupons that they are considered "undignified."}

I believe a careful examination of the empirical evidence would indicate that long-term borrowing and refunding are, to an important extent, postponable, thus allowing expectations to exert considerable influence. There is a tendency for a high level of interest rates to have a self-regulating influence on the supply of new long-term issues. During periods of relatively high long-term interest rates the Treasury has seldom floated long bonds despite its desire to lengthen the maturity of the debt.\footnote{During 1959 and early 1960, the Treasury was prevented from issuing securities with a maturity longer than five years because of the 4\% per cent interest ceiling on these issues.} "Poor" market conditions have also tended to remove many corporations from the capital markets. They may postpone coming to the market and may meet their immediate needs by drawing down their liquid assets or by short-term borrowings from commercial banks and other lenders. Similarly, state and local governments often have found it difficult to enter the long-term market during periods of stress, whereas they have responded to reduced levels of long-term rates by accelerating the financing of construction programs. Thus the introduction of expectations on the supply side of the market has the effect of dampening the fluctuations in long-term rates while causing additional pressures in the short-term sector as both the Treasury and corporations finance their needs with bills and certificates, on the one hand, and bank loans on the other.

The pressures on the bond market from both the demand and supply side are reinforcing in their effect. When interest rates are believed to be high in relation to historical precedent, investors will prefer to buy long-term bonds while issuers will prefer to sell short-term securities. Conversely, low interest rates will encourage investors to buy shorts and issuers to sell longs. We have opened, with care, the Pandora box of expectations and the direction, if not the magnitude, of their effect is unambiguous. Perhaps we may be charged with the Hicksian accusation that this leaves the rate of interest uncomfortably hanging by its own bootstraps.\footnote{Hicks, \textit{op. cit.}, p. 164.} We can only answer, with Mrs. Robinson, that the price today of any long-lived...
object with negligible carrying costs must be strongly influenced by expectations about what its price will be in the future.\footnote{Joan Robinson, *op. cit.*, p. 103.} If the rate of interest hangs by its own bootstraps, so does the price of a Miró painting.

**Concluding Remarks**

It will now be useful to summarize our findings and underline the advantages of the model which has been presented. First, we have shown that an explicit examination of theoretical bond price movements takes us a good part of the way toward understanding the observable structure of market interest rates. The inevitable tendency of the yield curve to flatten out as term to maturity is extended was explained and later demonstrated. Next we derived from our model an ascending and descending yield curve with behavioral assumptions far less demanding than in the Lutz theory. In fact, expectations were introduced only to the extent that future interest-rate fluctuations were anticipated to be contained within the range which has existed in the past. At a level of interest rates close to the upper end of the historical range we could explain a descending yield curve even where investors have formed no expectations as to the probable direction of interest-rate changes. When later we introduced stronger elements of foresight, our model had the advantage of being in closer conformity with the practices of bond investors who had always considered the Lutz theory chimerical. Furthermore, the Keynes-Hicks modifications were easily assimilated into the analytical framework. Finally, our model could be modified to take account of differences in yield which result from coupon differences, call features, tax advantages of discount bonds and institutional rigidities as they might apply to a particular maturity area. Thus, at least we have clothed the traditional expectational theory with new raiments which fit more closely the investing practices of bond investors and make the theory both simpler in its assumptions, and more tractable to modification. Hopefully, we have also gained added insights into the behavior of yields of securities with different terms to maturity.